

# A Class of Collisions of Plane Impulsive Light-Like Signals in General Relativity

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## Abstract

We present a systematic study of collisions of homogeneous, plane-fronted, impulsive light-like signals which do not interact after head-on collision. For the head-on collision of two such signals, six real parameters are involved, three from each of the incoming signals. We find two necessary conditions to be satisfied by these six parameters for the signals to be non-interacting after collision. We then solve the collision problem in general when these necessary conditions hold. After collision the two signals focus each other at Weyl curvature singularities on each others signal front. Our family of solutions contains some known collision solutions as special cases.

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# 1 Introduction

A few models in general relativity of the head-on collision of homogeneous, plane-fronted, impulsive light-like signals are known (see below) in which the signals do not interact after collision. This is in stark contrast to what happens after the head-on collision of homogeneous, plane-fronted, impulsive gravitational waves [1] in which back-scattered gravitational radiation appears after the collision and a curvature singularity develops. In this paper we make a systematic study of collisions of homogeneous, plane-fronted, impulsive light-like signals which do not interact after head-on collision. In such cases the region of the space-time model after collision is a portion of Minkowskian space-time and thus curvature singularity-free.

The space-time model of a general plane-fronted, impulsive, light-like signal is described by a line-element involving one function of three variables. Such a signal in general consists of a plane-fronted, light-like shell of matter accompanied by a plane-fronted, impulsive gravitational wave propagating through flat space-time. When specialised to be homogeneous this impulsive, light-like signal is characterised by three real parameters. One of these parameters is proportional to the relative energy density of the light-like shell of matter in the signal and the other two describe the two degrees of freedom of polarisation in the impulsive gravitational wave part of the signal. Thus for the head-on collision of two such signals, six real parameters are involved, three from each of the incoming signals. We find two necessary conditions to be satisfied by these six parameters for the signals to be non-interacting after collision. We then solve the collision problem in general when these necessary conditions hold. After collision the two signals focus each other at Weyl curvature singularities on each others signal front. Our family of solutions contains as special cases solutions found by Stoyanov [2], Babala [3] and Feinstein and Senovilla [4].

The paper is organised as follows: In section 2 the general plane-fronted, impulsive, light-like signal propagating through flat space-time and determined by a single function of three variables is described and specialised to the homogeneous case. In section 3 we set up the head-on collision problem, derive the two necessary conditions on the six parameters involved for no interaction between the signals after collision and then solve Einstein's vacuum field equations for the region of space-time after collision. Finally in section 4 properties of the family of collision solutions derived in section 3 are studied and contact is made with the known solutions mentioned above.

## 2 Plane Impulsive Light-Like Signal

The general plane-fronted impulsive light-like signal propagating through Minkowskian space-time can easily be constructed. Its history in otherwise flat space-time  $M$  is a null hyperplane  $\mathcal{N}$  on which the Riemann tensor has a Dirac delta function singularity and it incorporates in general a plane impulsive gravitational wave and a plane, light-like shell of matter having relative energy density, isotropic pressure and anisotropic stress. In coordinates covering both sides of  $\mathcal{N}$  the line-element of the space-time  $M$  is given by [5]

$$ds^2 = - \left( dx + \frac{u_+}{F_v} dF_x \right)^2 - \left( dy + \frac{u_+}{F_v} dF_y \right)^2 + 2 dv \left( du - \frac{u_+}{F_v} dF_v \right) . \quad (2.1)$$

Here  $u_+ = u \vartheta(u)$  with  $\vartheta(u)$  the Heaviside step function which is equal to unity if  $u > 0$  and is equal to zero if  $u < 0$ . The equation of  $\mathcal{N}$  is  $u = 0$  and we have  $u > 0$  to the future of  $\mathcal{N}$  while  $u < 0$  to the past of  $\mathcal{N}$ . The null geodesic integral curves of the vector field  $\partial/\partial v$  generate  $u = 0$  (and all other null hyperplanes  $u = \text{constant}$ ) and  $v$  is an affine parameter along these generators. The coordinates  $\xi^a = (x, y, v)$ , with  $a = 1, 2, 3$ , are intrinsic to the null hyperplanes  $u = \text{constant}$ . Also  $F = F(x, y, v)$  with partial derivatives indicated by subscripts and with  $F_v \neq 0$ . Calculation of the surface stress-energy tensor concentrated on  $u = 0$  shows that it describes light-like matter having relative energy density  $\epsilon$ , isotropic pressure  $P$  and anisotropic stress  $\Pi^{ab}$  given by [5]

$$\epsilon = -\frac{1}{8\pi} \frac{(F_{xx} + F_{yy})}{F_v} , \quad (2.2)$$

$$P = -\frac{1}{8\pi} \frac{F_{vv}}{F_v} , \quad (2.3)$$

$$\Pi^{12} = -\frac{1}{8\pi} \frac{F_{xv}}{F_v} , \quad \Pi^{13} = -\frac{1}{8\pi} \frac{F_{yv}}{F_v} , \quad (2.4)$$

with all other components of  $\Pi^{ab} = \Pi^{ba}$  vanishing. If any of these quantities are non-vanishing then the signal with history  $\mathcal{N}$  has a part which is a light-like shell of matter. The coefficient of  $\delta(u)$  in the Weyl tensor of  $M$  has a Petrov type N part, with degenerate principal null direction  $\partial/\partial v$ , given in the Newman-Penrose notation by

$$\hat{\Psi}_4 = \frac{1}{2F_v} (F_{xx} - F_{yy} - 2i F_{xy}) . \quad (2.5)$$

If this is non-vanishing then the signal with history  $\mathcal{N}$  has a part which is an impulsive gravitational wave. There is also a possibly non-zero Petrov type

II coefficient of  $\delta(u)$  in the Weyl tensor ( which if non-zero is associated with the presence of the light-like shell of matter described by (2.2)–(2.4)) given in Newman–Penrose notation by

$$\hat{\Psi}_2 = \frac{1}{3} \frac{F_{vv}}{F_v} , \quad \hat{\Psi}_3 = \frac{1}{\sqrt{2} F_v} (F_{xv} - iF_{yv}) . \quad (2.6)$$

We see that if the surface stress–energy of the light-like shell is isotropic in the sense that  $P$  and  $\Pi^{ab}$  in (2.3), (2.4) vanish, then the quantities in (2.6) vanish. The acid test therefore for the existence of a light-like shell part of the signal with history  $\mathcal{N}$  is the non-vanishing of some of (2.2)–(2.4), whereas the acid test for the existence of an impulsive gravitational wave part of the signal is the non-vanishing of (2.5). The calculation of (2.2)–(2.4) and of (2.5) involves independent sets of components of the jump in the transverse extrinsic curvature 3-tensor across  $\mathcal{N}$  (see [6]). This fact allows the decomposition of an impulsive light-like signal into the two distinct parts to be carried out unambiguously [7] [8] in the case of *any* impulsive light-like signal propagating through *any* space-time and not just in the case of the plane-fronted impulsive light-like signal propagating through Minkowskian space-time under consideration here.

The null geodesic integral curves of  $\partial/\partial u$  intersect the history  $\mathcal{N}$  of the impulsive light-like signal. The Newman–Penrose spin coefficients  $\sigma$  (the complex shear),  $\rho$  (the real expansion),  $\tau$  and  $\gamma$  associated with this null geodesic congruence undergo the following jumps (indicated by square brackets) across  $\mathcal{N}$ :

$$[\sigma] = -\hat{\Psi}_4^* , \quad [\rho] = 4\pi \epsilon , \quad [\tau] = 4\sqrt{2}\pi (\Pi^{12} + i\Pi^{13}) , \quad [\gamma] = 4\pi P , \quad (2.7)$$

with  $\hat{\Psi}_4^*$  the complex conjugate of  $\hat{\Psi}_4$  in (2.5) and  $\epsilon , \Pi^{12} , \Pi^{13} , P$  given in (2.2)–(2.4). We see Penrose’s observation [9] that a jump in the shear of the null geodesic congruence transverse to  $\mathcal{N}$  is necessary for the existence of a gravitational wave part to the signal and a jump in the expansion of this congruence is necessary for the shell part to have non-zero relative energy density.

The space-time  $M$  with line-element (2.1) can be constructed using a cut and paste technique in the style of Penrose [9] as follows: The line-element (2.1) with  $u > 0$  can be transformed to the manifestly flat form

$$ds_+^2 = -dx_+^2 - dy_+^2 + 2du_+dv_+ , \quad (2.8)$$

by the transformation [5]

$$x_+ = x + u \frac{F_x}{F_v} , \quad (2.9)$$

$$y_+ = y + u \frac{F_y}{F_v} , \quad (2.10)$$

$$v_+ = F + \frac{u}{2 F_v} (F_x^2 + F_y^2) , \quad (2.11)$$

$$u_+ = \frac{u}{F_v} . \quad (2.12)$$

Thus (2.8) is the line-element of  $M^+(u \geq 0)$ . The line-element (2.1) for  $u < 0$  can be transformed trivially into flat form

$$ds_-^2 = -dx_-^2 - dy_-^2 + 2 du_- dv_- , \quad (2.13)$$

by the tranformation

$$x_- = x , \quad y_- = y , \quad v_- = v , \quad u_- = u . \quad (2.14)$$

Thus (2.13) is the line-element of  $M^-(u \leq 0)$ . We see from (2.9)–(2.12) and (2.14) that on  $\mathcal{N}(u = 0)$  (which is clearly a null hyperplane),

$$x_+ = x_- , \quad y_+ = y_- , \quad v_+ = F(x_-, y_-, v_-) , \quad (2.15)$$

and these matching conditions leave the induced line-element on  $\mathcal{N}(u = 0)$  invariant:

$$dx_+^2 + dy_+^2 = dx_-^2 + dy_-^2 . \quad (2.16)$$

We have thus subdivided the space-time  $M$  into two halves  $M^+(u \geq 0)$  and  $M^-(u \leq 0)$ , each with boundary  $\mathcal{N}(u = 0)$ , and we have then reattached the halves on  $\mathcal{N}$  with the mapping (2.15) which preserves the induced line-element on  $\mathcal{N}$ .

For a homogeneous signal we take

$$F(x, y, v) = v - \frac{a}{2} (x^2 + y^2) + \frac{b}{2} (x^2 - y^2) + c x y , \quad (2.17)$$

with  $a, b, c$  constants and  $a \geq 0$ . This signal has an isotropic stress-energy ( $P = \Pi^{12} = \Pi^{13} = 0$ ) with relative energy density (2.2) taking the form

$$\epsilon = \frac{a}{4\pi} , \quad (2.18)$$

and is accompanied by an impulsive gravitational wave with

$$\hat{\Psi}_4 = b - i c , \quad (2.19)$$

having the maximum two degrees of freedom of polarisation. The line-element (2.1) reduces in this case to

$$ds^2 = - \{ (1 - (a - b) u_+) dx + c u_+ dy \}^2 - \{ c u_+ dx + (1 - (a + b) u_+) dy \}^2 + 2 du dv . \quad (2.20)$$

This line-element has the Rosen–Szekeres form

$$ds^2 = -e^{-U} \left( e^V \cosh W dx^2 - 2 \sinh W dx dy + e^{-V} \cosh W dy^2 \right) + 2 e^{-M} du dv , \quad (2.21)$$

with

$$e^{-U} = 1 - 2a u_+ + (a^2 - b^2 - c^2) u_+^2 , \quad (2.22)$$

$$e^V = \left[ \frac{(1 - (a - b) u_+)^2 + c^2 u_+^2}{(1 - (a + b) u_+)^2 + c^2 u_+^2} \right]^{\frac{1}{2}} , \quad (2.23)$$

$$\sinh W = \frac{-2c u_+ (1 - a u_+)}{1 - 2a u_+ + (a^2 - b^2 - c^2) u_+^2} , \quad (2.24)$$

$$M = 0 . \quad (2.25)$$

In this paper we shall consider the head-on collision of this plane-fronted light-like signal with a signal of similar type. This latter signal is described by a space-time with line-element (2.21) with

$$e^{-U} = 1 - 2\alpha v_+ + (\alpha^2 - \beta^2 - \gamma^2) v_+^2 , \quad (2.26)$$

$$e^V = \left[ \frac{(1 - (\alpha - \beta) v_+)^2 + \gamma^2 v_+^2}{(1 - (\alpha + \beta) v_+)^2 + \gamma^2 v_+^2} \right]^{\frac{1}{2}} , \quad (2.27)$$

$$\sinh W = \frac{-2\gamma v_+ (1 - \alpha v_+)}{1 - 2\alpha v_+ + (\alpha^2 - \beta^2 - \gamma^2) v_+^2} , \quad (2.28)$$

$$M = 0 , \quad (2.29)$$

where  $\alpha, \beta, \gamma$  are real constants,  $\alpha \geq 0$ , and  $v_+ = v \vartheta(v)$ . Here the history of the light-like signal is the null hyperplane  $v = 0$ .

### 3 Interaction-Free Signals After Collision

The space-time model of the head-on collision of two plane-fronted, homogeneous, impulsive, light-like signals will have a line-element of the form (2.21) with, in general,  $U, V, W$  and  $M$  functions of  $u, v$ . In the region  $v < 0, u > 0$  the functions  $U, V, W$  and  $M$  are given by (2.22)–(2.25) with  $u_+ = u$  and in the region  $u < 0, v > 0$  these functions are given by (2.26)–(2.29) with  $v_+ = v$ . The region  $u < 0, v < 0$  has line-element (2.21) with  $U = V = W = M = 0$ , which is consistent with (2.22)–(2.29) when both  $u < 0$  and  $v < 0$ . The line-element in the region  $u > 0, v > 0$  (after the collision) has the form (2.21) with  $U, V, W$  and  $M$  functions of  $u, v$  satisfying the O’Brien–Synge [10] junction conditions: For  $v = 0, u > 0$  the functions  $U, V, W$  and  $M$  are given by

(2.22)–(2.25) with  $u_+ = u$  and for  $u = 0, v > 0$  the functions  $U, V, W$  and  $M$  are given by (2.26)–(2.29) with  $v_+ = v$ . Einstein's vacuum field equations are to be solved for  $U, V, W$  and  $M$  in the region  $u > 0, v > 0$  (after the collision) subject to these boundary (junction) conditions. These equations are [11] (with subscripts denoting partial derivatives)

$$U_{uv} = U_u U_v , \quad (3.1)$$

$$2V_{uv} = U_u V_v + U_v V_u - 2(V_u W_v + V_v W_u) \tanh W , \quad (3.2)$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W , \quad (3.3)$$

$$2U_u M_u = -2U_{uu} + U_u^2 + W_u^2 + V_u^2 \cosh^2 W , \quad (3.4)$$

$$2U_v M_v = -2U_{vv} + U_v^2 + W_v^2 + V_v^2 \cosh^2 W , \quad (3.5)$$

$$2M_{uv} = -U_{uv} + W_u W_v + V_u V_v \cosh^2 W . \quad (3.6)$$

The first of these equations can immediately be solved [13] in conjunction with the boundary conditions to be satisfied by  $U$  on  $u = 0, v > 0$  and on  $v = 0, u > 0$  to yield, in  $u > 0, v > 0$ ,

$$e^{-U} = 1 - 2a u - 2\alpha v + (a^2 - b^2 - c^2) u^2 + (\alpha^2 - \beta^2 - \gamma^2) v^2 . \quad (3.7)$$

The problem is to solve (3.2), (3.3) for  $V, W$  subject to the boundary conditions and then to solve (3.4) and (3.5) for  $M$ . Equation (3.6) is the integrability condition for (3.4) and (3.5).

We can have non-interacting signals after collision by requiring that the region  $u > 0, v > 0$  be, if possible, a portion of Minkowskian space-time. Necessary conditions for this to be possible can be obtained by examining approximate solutions of the vacuum field equations, subject to our boundary conditions, in the region  $u > 0, v > 0$  near  $u = 0$  (i.e. for small  $v > 0$ ) and/or near  $v = 0$  (i.e. for small  $u > 0$ ). In this regard it is helpful to note that the non-identically vanishing components of the Riemann curvature tensor for the line-element (2.21) are given in Newman–Penrose notation by [11]

$$2\Psi_0 = B_v + (M_v - U_v) B + iB V_v \sinh W , \quad (3.8)$$

$$2\Psi_2 = M_{uv} - \frac{1}{4} (A \bar{B} - \bar{A} B) , \quad (3.9)$$

$$2\Psi_4^* = A_u + (M_u - U_u) A + iA V_u \sinh W , \quad (3.10)$$

with

$$A = -V_u \cosh W + iW_u , \quad B = -V_v \cosh W + iW_v , \quad (3.11)$$

and the star, as before, denotes complex conjugation. Clearly necessary, but not necessarily sufficient, conditions for flatness in the region of space-time  $u > 0, v > 0$  are

$$M_{uv} = 0 , \quad (3.12)$$

and

$$A \bar{B} - \bar{A} B = 0 . \quad (3.13)$$

The vacuum field equations (3.1) and (3.6) imply that (3.12) is equivalent to

$$A \bar{B} + \bar{A} B = 2 U_u U_v . \quad (3.14)$$

The conditions (3.13) and (3.14) are to hold throughout the region  $u > 0, v > 0$  and in particular near  $u = 0$  and/or  $v = 0$  in this region. It will be sufficient for our immediate purpose to calculate (3.13) and (3.14) at  $u = v = 0$  (strictly speaking in the limit  $u \rightarrow 0^+$  and  $v \rightarrow 0^+$ ). If *when*  $v = 0$  we put

$$A = S e^{i\psi} , \quad B = R e^{i\phi} , \quad (3.15)$$

we have immediately from our boundary conditions on  $v = 0$  ( $u > 0$ )

$$S = \frac{2 \sqrt{b^2 + c^2}}{(1 - a u)^2 - (b^2 + c^2) u^2} , \quad (3.16)$$

and  $\psi = \theta + \pi$  with

$$e^{i\theta} = \frac{b\{(1 - a u)^2 - (b^2 + c^2) u^2\} + i c \{(1 - a u)^2 + (b^2 + c^2) u^2\}}{\sqrt{b^2 + c^2} \mathcal{F} \mathcal{G}} , \quad (3.17)$$

with

$$\mathcal{F} = \sqrt{(1 - (a - b) u)^2 + c^2 u^2} , \quad \mathcal{G} = \sqrt{(1 - (a + b) u)^2 + c^2 u^2} . \quad (3.18)$$

It is useful to note that

$$\theta_u = \frac{4 b c u (1 - a u)}{\mathcal{F}^2 \mathcal{G}^2} . \quad (3.19)$$

The field equations (3.2) and (3.3) evaluated on  $v = 0$  give us the following differential equations for  $R, \phi$ :

$$R_u = \frac{\{a - (a^2 - b^2 - c^2) u\} R}{(1 - a u)^2 - (b^2 + c^2) u^2} + \frac{2 \alpha \sqrt{b^2 + c^2} \cos(\psi - \phi)}{\{(1 - a u)^2 - (b^2 + c^2) u^2\}^2} , \quad (3.20)$$

$$\phi_u = \frac{4 b c u (1 - a u)}{\mathcal{F}^2 \mathcal{G}^2} + \frac{2 \alpha \sqrt{b^2 + c^2} \sin(\psi - \phi)}{R \{(1 - a u)^2 - (b^2 + c^2) u^2\}^2} . \quad (3.21)$$

Writing  $\hat{R} = R \sqrt{(1 - a u)^2 - (b^2 + c^2) u^2}$  and using (3.19) these can be simplified to

$$\hat{R}_u = \frac{2 \alpha \sqrt{b^2 + c^2} \cos(\psi - \phi)}{\{(1 - a u)^2 - (b^2 + c^2) u^2\}^{\frac{3}{2}}} , \quad (3.22)$$

$$(\psi - \phi)_u = \frac{-2 \alpha \sqrt{b^2 + c^2} \sin(\psi - \phi)}{\hat{R} \{(1 - a u)^2 - (b^2 + c^2) u^2\}^{\frac{3}{2}}} . \quad (3.23)$$

These give us

$$\hat{R} \sin(\psi - \phi) = K , \quad (3.24)$$

$$\hat{R} \cos(\psi - \phi) = \frac{2\alpha \{a - (a^2 - b^2 - c^2)u\}}{\sqrt{b^2 + c^2} \sqrt{(1 - au)^2 - (b^2 + c^2)u^2}} + C , \quad (3.25)$$

where  $K, C$  are constants of integration. From our boundary conditions we have, when  $u = v = 0$ ,

$$e^{i\psi} = -\frac{(b + ic)}{a} , \quad e^{i\phi} = -\frac{(\beta + i\gamma)}{\alpha} , \quad R = 2\alpha . \quad (3.26)$$

This determines the constants  $K, C$  and writing  $\hat{R}$  in terms of  $R$  we have from (3.24) and (3.25)

$$R \sin(\psi - \phi) = \frac{2(\beta c - \gamma b)}{\sqrt{b^2 + c^2} \sqrt{(1 - au)^2 - (b^2 + c^2)u^2}} , \quad (3.27)$$

$$R \cos(\psi - \phi) = \frac{2\alpha \{a - (a^2 - b^2 - c^2)u\}}{\sqrt{b^2 + c^2} \sqrt{(1 - au)^2 - (b^2 + c^2)u^2}} + \frac{2(\beta b + \gamma c - \alpha a)}{\sqrt{b^2 + c^2} \sqrt{(1 - au)^2 - (b^2 + c^2)u^2}} . \quad (3.28)$$

Now (3.13) evaluated at  $v = 0$  gives

$$\sin(\psi - \phi) = 0 , \quad (3.29)$$

and by (3.27) this is equivalent to

$$\beta c - \gamma b = 0 . \quad (3.30)$$

Next using  $U$  given by (3.7),  $S$  given by (3.16) and  $R \cos(\psi - \phi)$  by (3.28), the equation (3.14) evaluated at  $v = 0$  is found to be equivalent to

$$\beta b + \gamma c - \alpha a = 0 . \quad (3.31)$$

Hence *the relations (3.30) and (3.31) between the three real constants  $a, b, c$  associated with one in-coming light-like signal and the three real constants  $\alpha, \beta, \gamma$  associated with the second incoming light-like signal are necessary conditions for the region of space-time  $u > 0, v > 0$  (after the collision) to be flat.* We note that (3.30) and (3.31) are invariant under the interchange of  $a, b, c$  with  $\alpha, \beta, \gamma$  respectively and so would also emerge had we evaluated (3.13) and (3.14) at  $u = 0$  instead of at  $v = 0$ . We note that (3.30) and (3.31)

exclude the case of a collision of two plane impulsive gravitational waves [1] (corresponding to  $b \neq 0, \beta \neq 0, a = \alpha = \gamma = c = 0$ ) and also a collision of two plane light-like shells [12] (corresponding to  $a \neq 0, \alpha \neq 0, b = c = \beta = \gamma = 0$ ). In those two cases the spacetime after the collision is not flat and contains a curvature singularity.

We shall henceforth assume that the conditions (3.30) and (3.31) are satisfied and we proceed to solve the vacuum field equations (3.2)–(3.6) in the region  $u > 0, v > 0$  with the boundary conditions on  $U, V, W$  and  $M$  indicated in the opening paragraph of this section. We use a procedure described in [14] to which we must refer the reader for details. The solution derived can of course be verified by substitution into the field equations. The preliminary calculations necessary to implement the procedure in [14] have been carried out above. These are the calculations determining  $S, \psi, R, \phi$  in (3.15). The results are given by (3.16), (3.17), (3.27) and (3.28) with (3.30) and (3.31) now holding. Using these as starting point we find that, in addition to  $U$  already given in  $u > 0, v > 0$  by (3.7), we obtain

$$e^V = \left[ \frac{\{1 - (a - b)u - (\alpha - \beta)v\}^2 + (cu + \gamma v)^2}{\{1 - (a + b)u - (\alpha + \beta)v\}^2 + (cu + \gamma v)^2} \right]^{\frac{1}{2}}, \quad (3.32)$$

$$\sinh W = -2(cu + \gamma v)(1 - au - \alpha v)e^U, \quad (3.33)$$

$$M = 0. \quad (3.34)$$

If in these functions we replace  $u$  by  $u_+ = u \vartheta(u)$  and  $v$  by  $v_+ = v \vartheta(v)$  then we can include in a single expression in each case the form of the function in the four different regions of the collision space-time. Having done this and having substituted the functions into the line-element (2.21) we arrive at the final form of our line-element:

$$\begin{aligned} ds^2 = & -\{(1 - (a - b)u_+ - (\alpha - \beta)v_+)dx + (cu_+ + \gamma v_+)dy\}^2 \\ & -\{(cu_+ + \gamma v_+)dx + (1 - (a + b)u_+ - (\alpha + \beta)v_+)dy\}^2 \\ & + 2du dv, \end{aligned} \quad (3.35)$$

with the six parameters  $a, b, c, \alpha, \beta, \gamma$  satisfying (3.30) and (3.31). The non-identically vanishing Weyl tensor components for the space-time with line-element (3.35) are, in Newman–Penrose notation,

$$\Psi_0 = -\frac{\alpha \{a - (a^2 - b^2 - c^2)u_+\} \mathcal{P}}{(b^2 + c^2) \mathcal{F} \mathcal{G} \mathcal{P}_1} \delta(v), \quad (3.36)$$

$$\Psi_4 = -\frac{a \{\alpha - (\alpha^2 - \beta^2 - \gamma^2)v_+\} \mathcal{Q}}{(\beta^2 + \gamma^2) \mathcal{F}' \mathcal{G}' \mathcal{Q}_1} \delta(u). \quad (3.37)$$

Here

$$\mathcal{P} = b \mathcal{P}_1 + i c \mathcal{P}_2 , \quad (3.38)$$

$$\mathcal{Q} = \beta \mathcal{Q}_1 - i \gamma \mathcal{Q}_2 , \quad (3.39)$$

$$\mathcal{P}_1 = \{(1 - a u_+)^2 - (b^2 + c^2) u_+^2\} , \quad (3.40)$$

$$\mathcal{P}_2 = \{(1 - a u_+)^2 + (b^2 + c^2) u_+^2\} , \quad (3.41)$$

$$\mathcal{Q}_1 = \{(1 - \alpha v_+)^2 - (\beta^2 + \gamma^2) v_+^2\} , \quad (3.42)$$

$$\mathcal{Q}_2 = \{(1 - \alpha v_+)^2 + (\beta^2 + \gamma^2) v_+^2\} , \quad (3.43)$$

$$\mathcal{F}' = \sqrt{(1 - (\alpha - \beta) v_+)^2 + \gamma^2 v_+^2} , \quad (3.44)$$

$$\mathcal{G}' = \sqrt{(1 - (\alpha + \beta) v_+)^2 + \gamma^2 v_+^2} , \quad (3.45)$$

and  $\mathcal{F}, \mathcal{G}$  are given by (3.18) with  $u$  replaced by  $u_+$ . Labelling the coordinates  $x^\mu = (x, y, u, v)$  with  $\mu = 1, 2, 3, 4$  the Ricci tensor components for the space-time with line-element (3.35) are

$$R_{\mu\nu} = -\frac{2\alpha}{\mathcal{P}_1} \delta(v) l_\mu l_\nu - \frac{2a}{\mathcal{Q}_1} \delta(u) n_\mu n_\nu , \quad (3.46)$$

with  $l^\mu = \delta_3^\mu$  and  $n^\mu = \delta_4^\mu$ . Finally we note that on  $v = 0$  the (real) expansion  $\rho_l$  and the modulus of the complex shear  $|\sigma_l|$  of the null geodesic integral curves of  $l^\mu$  are given by

$$\rho_l = \frac{a \vartheta(u) - (a^2 - b^2 - c^2) u_+}{\mathcal{P}_1} , \quad |\sigma_l| = \frac{\sqrt{b^2 + c^2} \vartheta(u)}{\mathcal{P}_1} , \quad (3.47)$$

while on  $u = 0$  the (real) expansion  $\rho_n$  and the modulus of the complex shear  $|\sigma_n|$  of the null geodesic integral curves of  $n^\mu$  are given by

$$\rho_n = \frac{\alpha \vartheta(v) - (\alpha^2 - \beta^2 - \gamma^2) v_+}{\mathcal{Q}_1} , \quad |\sigma_n| = \frac{\sqrt{\beta^2 + \gamma^2} \vartheta(v)}{\mathcal{Q}_1} . \quad (3.48)$$

## 4 Discussion

We remark that for the space-time with line-element (3.35) it follows from (3.36), (3.37) and (3.46) that the region  $u > 0, v > 0$  after the collision is flat. To see this explicitly we consider two cases separately.

I ( $c = 0 = \gamma$ ) : for  $u > 0, v > 0$  we have

$$ds^2 = -\{1 - (a - b)u - (\alpha - \beta)v\}^2 dx^2 - \{1 - (a + b)u - (\alpha + \beta)v\}^2 dy^2 + 2 du dv , \quad (4.1)$$

with  $a\alpha - b\beta = 0$ . We first note that if  $a = \pm b$  ( $\Leftrightarrow \alpha \pm \beta$ ) then it is easy to transform (4.1) to Minkowskian form [15]. Now assume  $a \neq \pm b$  ( $\Leftrightarrow \alpha \neq \pm\beta$ ) and make the transformation

$$u' = 1 - (a - b)u - (\alpha - \beta)v, \quad (4.2)$$

$$v' = 1 - (a + b)u - (\alpha + \beta)v. \quad (4.3)$$

This results in (4.1) taking the form

$$ds^2 = -u'^2 dx^2 - v'^2 dy^2 - \lambda^2 du'^2 + \mu^2 dv'^2, \quad (4.4)$$

with  $\lambda^{-2} = -2(\alpha - \beta)(a - b)$ ,  $\mu^{-2} = 2(\alpha + \beta)(a + b)$ . We note that  $(\alpha - \beta)(a - b) < 0$  and also  $(\alpha^2 - \beta^2)(a^2 - b^2) = -(\alpha b - \beta a)^2$  and so  $(\alpha + \beta)(a + b) > 0$ . Then with  $\bar{u} = \lambda u'$ ,  $\bar{v} = \mu v'$ ,  $\bar{x} = \lambda^{-1}x$ ,  $\bar{y} = \mu^{-1}y$  we have

$$ds^2 = -\bar{u}^2 d\bar{x}^2 - \bar{v}^2 d\bar{y}^2 - d\bar{u}^2 + d\bar{v}^2. \quad (4.5)$$

Now put  $X = \bar{u} \cos \bar{x}$ ,  $Y = \bar{u} \sin \bar{x}$ ,  $Z = \bar{v} \sinh \bar{y}$ ,  $T = \bar{v} \cosh \bar{y}$  and we arrive at

$$ds^2 = -dX^2 - dY^2 - dZ^2 + dT^2. \quad (4.6)$$

We note that  $0 \leq \bar{x} < 2\pi$  and so  $0 \leq x < 2\pi\lambda$ . This periodicity in the coordinate  $x$  has been noted already in a special case [3] of (3.35) which we draw attention to below.

II( $c \neq 0 \neq \gamma$ ): for  $u > 0, v > 0$  in this general case we have from (3.35)

$$\begin{aligned} ds^2 = & -\{(1 - (a - b)u - (\alpha - \beta)v)dx + (cu + \gamma v)dy\}^2 \\ & -\{(cu + \gamma v)dx + (1 - (a + b)u - (\alpha + \beta)v)dy\}^2 \\ & + 2dudv, \end{aligned} \quad (4.7)$$

with  $a\alpha - b\beta = c\gamma$  and  $\beta c = b\gamma$ . The rotation

$$x = \frac{x' + \Lambda y'}{\sqrt{1 + \Lambda^2}}, \quad y = \frac{-\Lambda x' + y'}{\sqrt{1 + \Lambda^2}}, \quad (4.8)$$

with  $\Lambda$  satisfying

$$\Lambda^2 - 2\frac{b}{c}\Lambda - 1 = 0, \quad (4.9)$$

results in (4.7) taking the form

$$ds^2 = -\{1 - (a' - b')u - (\alpha' - \beta')v\}^2 dx'^2 - \{1 - (a' + b')u - (\alpha' + \beta')v\}^2 dy'^2 + 2dudv, \quad (4.10)$$

with

$$a' = a, \quad b' = b - \Lambda c, \quad \alpha' = \alpha, \quad \beta' = \beta - \Lambda \gamma, \quad (4.11)$$

from which it follows that

$$a'\alpha' - b'\beta' = 0 . \quad (4.12)$$

Now (4.10) with (4.12) is identical to the case I considered above. It thus follows that (4.10) (and hence (4.7)) can be written in manifestly Minkowskian form and that in this case  $x'$  is a periodic coordinate.

The special case of (3.35) corresponding to  $a = b, \alpha = \beta$  and  $c = \gamma = 0$  gives a subset of a family of solutions originally found by Stoyanov [2]. The solution (3.35) in this case describes the collision of two linearly polarised plane impulsive gravitational waves each sharing its wave front with a plane light-like shell with relative energy density proportional to the amplitude of the wave. The special case of (3.35) corresponding to  $a \neq 0, b = 0, \alpha = 0, \beta \neq 0, c = \gamma = 0$ , describes the collision of a linearly polarised plane impulsive gravitational wave with a plane light-like shell and was found by Babala [3]. The generalisation of this for which  $a \neq 0, b = c = 0, \alpha = 0, \beta \neq 0, \gamma \neq 0$  was found by Feinstein and Senovilla [4].

We see from (3.47) and (3.48) that the signals involved in the collision focus each other after collision on their signal fronts at  $v = 0, \mathcal{P}_1 = 0$  and at  $u = 0, \mathcal{Q}_1 = 0$ , where it follows from (3.36) and (3.37) that the Weyl curvature is singular and from (3.46) that the surface stress-energy of the light-like shells (after collision) is singular in each case. On the signal front  $u = 0, v > 0$  or  $v = 0, u > 0$  the signals are focused on two cylinders with elliptic cross-sections in general. It is important to note in this regard that when  $c = 0$  we have  $\mathcal{F}\mathcal{G} = 0 \Leftrightarrow \mathcal{P}_1 = 0$  and similarly when  $\gamma = 0$  we have  $\mathcal{F}'\mathcal{G}' = 0 \Leftrightarrow \mathcal{Q}_1 = 0$ . Also since the coordinates  $x$  in case I above and  $x'$  in case II are periodic the space-time region  $u > 0, v > 0$  does not extend to infinity but fills a cylinder with, in general, elliptic cross-sections. A topological singularity occurs on  $(a + \sqrt{b^2 + c^2})u + (\alpha + \sqrt{\beta^2 + \gamma^2})v = 1$ , and the space-time cannot be uniquely extended beyond this singularity. For an exceptional choice of the parameters (see [3], for example) the two cylinders on which the signals focus each other degenerate into a point and a circle.

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$$-dx^2 - (1 - 2au - 2\alpha v)^2 dy^2 + 2du dv = -dx^2 - dY^2 - dz^2 + dt^2 ,$$

with  $Y = \frac{(1-2au-2\alpha v)}{\sqrt{8a\alpha}} \sinh(\sqrt{8a\alpha} y)$ ,  $z = \frac{1+2au-2\alpha v}{\sqrt{8a\alpha}}$  and  $t = \frac{(1-2au-2\alpha v)}{\sqrt{8a\alpha}} \cosh(\sqrt{8a\alpha} y)$  and we note that for physical reasons  $a > 0$  and  $\alpha > 0$ . The case  $a = -b$  ( $\Leftrightarrow \alpha = -\beta$ ) is the same as this with  $x$  and  $y$  interchanged.